Option pricing in a hidden Markov model of the short rate

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Regime Switching short-rates

Regime-switching behaviour of an economy was first reported by J. Hamilton (1989).

Bonds

Options

Derivatives pricing in a stochastic rate economy: Merton (1973)

Most papers assume regime switching behavior in the underlying asset.

Our Result: a simple formula for pricing (european) options when:

- assets have a diffusive behavior;

- discounting factors display sudden jumps due to a shift of the short rate from one regime to another one.
Basic example: Vasicek model with two long-term means


\[ dr_t = \kappa(\theta(X_t) - r_t)dt + \sigma(r)dW^{(r)}(t), \]

\[ X_t = \text{discrete-state, continuous time Markov Chain}. \]

The term structure \( p(t, T) \) is said **semi-affine** if

\[ p(t, T) \equiv p(t, T|r_t = r, X_t = i) = e^{A_i(t,T) - B_i(t,T)r}. \]

(here \( A \) and \( B \) can depend on the state \( i \))
Case of two means:

- \( i \in \{1, 2\} \) and \( \theta(X_t) \in \{\theta_1, \theta_2\} \)

- \( B_1 = B_2 \equiv B(t, T) = (1 - \exp(-\kappa(T - t))) / \kappa \)

and \( A_1 \equiv A_1(t, T), A_2 \equiv A_2(t, T) \) must solve

\[
\begin{align*}
\frac{\partial}{\partial t} A_1 - \kappa \theta_1 B + \frac{1}{2} (\sigma(r) B)^2 + h_T (e^{-(A_2 - A_1)} - 1) &= 0 \\
\frac{\partial}{\partial t} A_2 - \kappa \theta_2 B + \frac{1}{2} (\sigma(r) B)^2 + h_T (e^{(A_2 - A_1)} - 1) &= 0,
\end{align*}
\]

Boundary conditions: \( A_1(T, T) = A_2(T, T) = 0 \)

The term-structure model

\[
\begin{aligned}
\frac{dr_t}{dt} &= a(t, r_t, X_t)dt + b'(t, r_t, X_t)dW^{(r)}(t), \\
\frac{dX_t}{dt} &= \gamma(t, X_t)dt + \xi(t, X_t)dW^{(r)}(t) + \int_E \delta(t, X_{t-}, z) \mu(dt, dz)
\end{aligned}
\]

- \( a \in \mathbb{R}, \ b \in \mathbb{R}^d, \ \gamma \in \mathbb{R}^m \) and \( \xi \in \mathbb{R}^{m \times d} \) satisfy standard regularity conditions for existence and pathwise uniqueness of the SDE;

- \( \mu(dt, dx) \) is a marked point process on a measurable Lusin mark space \((E, \mathcal{E})\);

- the \( Q \)-compensator \( \nu(\omega; dt, dz) \) of \( \mu \) has a predictable intensity of the form \( \lambda_t(\omega; dz)dt = \lambda(t, X(t-, \omega), dz)dt \);

- the Wiener process and the marked point process are independent.
The price $p(t, T)$ at time $t$ of a zero coupon bond maturing at $T \geq t$ can be obtained by an application of the generalized Ito’s formula:

$$dp(t, T) = p(t, T)\left\{ \eta^{(p)}(t, T, r_t, X_t)dt + \sigma^{(p)}(t, T, r_t, X_t)'dW^{(r)}(t) \right\}$$

$$+ \int_E p(t-, T)[e^{\Delta A(t,T,X,t-,z)} - 1]\mu(dt, dz)$$

with $\eta^{(p)} \in \mathbb{R}$ and $\sigma^{(p)} \in \mathbb{R}^d$ given by

- $\eta^{(p)}(t, T, r, x) = r - \int_E e^{\Delta A(t,T,x,z)} - 1\lambda(t, x, dz)$ (drift);

- $\sigma^{(p)}(t, T, r, x) = \xi' D_x A - bB$ (volatility);

- $\Delta A(t, T, x, z) \equiv A(t, T, x + \delta(t, x, z)) - A(t, T, x)$ (A - jump).
The underlying asset price process is
\[ dS(t) = S(t)[r_t dt + \sigma^{(s)}(t)dW^{(s)}(t)] \]
- \( \sigma^{(s)}(t) \) is a positive deterministic function;
- \( W^{(s)} \in \mathbb{R} \) is defined on the same stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\) which carries the Wiener process \( W^{(r)} \in \mathbb{R}^d \);
- the two processes are correlated:
\[
W^{(r)}(t) = W^{(1)}(t) \\
W^{(s)}(t) = \rho'W^{(1)}(t) + \sqrt{1 - \rho'^2}W^{(2)}(t)
\]
where \( \rho' = (\rho_1, \ldots, \rho_d) \in [-1, 1]^d \) is the correlation vector and \( W^{(1)}(t) \in \mathbb{R}^d \) and \( W^{(2)}(t) \in \mathbb{R} \) are independent standard Wiener processes.
The $T$-forward measure is defined to be a probability measure $Q_T$, absolutely continuous with respect to $Q$, under which the process

$$Z(t; T) = \frac{S(t)}{p(t, T)}$$

is a martingale w.r.t. the natural filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ generated by the processes $W^{(1)}(\cdot), W^{(2)}(\cdot)$ and $\mu(\cdot, \cdot)$.

The price at time $t$ of an option with maturity $T$, $\Pi(t, T)$, with terminal pay-off $\Phi(S(T))$ (having finite expectation w.r.t. $Q_T$) is given by

$$\Pi(t, T) = p(t, T) \mathbb{E}^T [\Phi(Z(T; T))|\mathcal{G}_t]$$

where $\mathbb{E}^T$ is the expectation w.r.t. the $Q_T$ measure.
Fact 1

Suppose \((Hp)\) \(\sigma^{(p)}\) is a deterministic function and some no-arbitrage conditions hold.

Then it can be shown that forward measures \(Q^T\) (equivalent to \(Q\)) do exist for these models and \(Z(t;T)\) are \(Q^T\)-martingales.

Fact 2

\(\log Z(t;T)\) can be decomposed as

\[
\underbrace{Y(t,T)}_{\text{jump part}} + \mathcal{N}(\log Z(t; t) - \frac{1}{2} \int_t^T \omega^2(s, T)ds, \int_t^T \omega^2(s, T)ds)
\]

The form of \(Y(t,T)\) can be explicitly given.
Main result (Ramponi, Scarlatti (2005))

(i) Suppose $\mathbb{E}^T(e^Y(t,T))$ is finite (satisfied in many models).

(ii) Suppose $Y(t, T)$ and $\mathcal{N}$ are independent under $Q^T$ (satisfied in some models).

Then

$$c(t, T; K) = \mathbb{E}^T[e^Y(t,T)c^{SM}(t, T; Ke^{-Y(t,T)})]$$

$$= S(t)\mathbb{E}^T[e^Y(t,T)N[d_1(Ke^{-Y(t,T)})] - Kp(t, T)\mathbb{E}^T[N[d_2(Ke^{-Y(t,T)})]],$$

where

$$c^{SM}(t, T; \chi) = S(t)N[d_1(\chi)] - \chi p(t, T)N[d_2(\chi)]$$
is the Merton’s pricing formula with bond value \( p(t, T) \) given by the semi-affine term structure and

\[
d_2(\chi) = \left( \log\left( \frac{S(t)}{\chi p(t, T)} \right) - \frac{1}{2} \Omega^2(t, T) \right) / (\sqrt{\Omega^2(t, T)}),
\]
\[
d_1(\chi) = d_2(\chi) + \sqrt{\Omega^2(t, T)}, \quad \Omega^2(t, T) = \int_t^T \omega^2(s, T) \, ds,
\]
\[
\omega(t, T) = \text{volatility of } Z(t, T).
\]

**Remarks.**

Monte Carlo sampling to calculate \( c(t, T; K) \).

When \( Y(t, T) \equiv 0 \) a.e. our formula reduces to the original Merton’s formula.

In the case of two long-term means, this happens for

\[
\theta_1 = \theta_2 \equiv \theta
\]
Valuation of corporate risky debts

Following Merton:

- \( S(t) \) represents the firm’s asset value

- default can occur only when the firm cannot repay at time \( T \) the promised principal, denoted by \( F \).

The value \( D(t, T) \) of a corporate discount bond with maturity \( T \) can be evaluated by the following formula

\[
D(t, T) = p(t, T) \{ Z^T(t) (1 - \mathbf{E}^T [e^{Y(t,T)} N[d_1(F e^{-Y(t,T)})]]) \\
+ F \mathbf{E}^T [N[d_2(F e^{-Y(t,T)})]]} \}.
\]
Valuation of corporate risky debts: credit spreads

Credit spreads

\[ \delta(t, T) = -\frac{1}{T-t} \log \left( \frac{D(t, T)}{Fp(t, T)} \right) \]

have the following expression (\( k(t, T) = \frac{F}{Z^T(t)} = \text{firm's leverage} \))

\[
\begin{align*}
\delta(t, T) &= -\frac{1}{T-t} \log \left( (k(t, T))^{-1} \left( 1 - \mathbf{E}^T \left[ e^{Y(t,T)} N[d_1(F e^{-Y(t,T)})] \right] \right) \right) \\
&\quad + \mathbf{E}^T \left[ N[d_2(F e^{-Y(t,T)})] \right].
\end{align*}
\]
Pricing bond options

Same technique as before:

\[ c(t, T_1; K) = \mathbb{E}^{T_1}[e^{Y(t,T_1,T_2)}c^{SJ}(t, T_1; Ke^{-Y(t,T_1,T_2)})] \]

where

\[ c^{SJ}(0, T_1; K) = p(t, T_2)N[d_1(K)] - Kp(t, T_1)N[d_2(K)] \]

is the Jamshidian’s call formula with the bond value \( p(t, \cdot) \) given by the semi-affine term structure and

\[ d_2(\chi) = (\log(\frac{p(t,T_2)}{\chi p(t,T_1)}) - \frac{1}{2}\Omega^2(t, T_1, T_2)) / (\sqrt{\Omega^2(t, T_1, T_2)}) \]

\[ d_1(\chi) = d_2(\chi) + \sqrt{\Omega^2(t, T_1, T_2)}, \quad \Omega^2(t, T_1, T_2) = \int_t^{T_1} \sigma^2(s, T_1, T_2)ds \]

and \( \mathbb{E}^{T_1}[\cdot] \) denotes expectation w.r.t. \( Q^{T_1} \).
Some properties of call prices for basic example

Let $g_1$ and $g_2$ be the two possible positive levels of mean reversion for the short rate and $\epsilon = g_1 - g_2 \geq 0$.

For $\epsilon = 0$ $Y_i(t, T) = 0$ w.p.1 and $p(t, T) = p^V_\theta(t, T)$.

The price of a call option is given by (Merton)

$$c^V_M(t, T; K) = S(t)N[d_1(K)] - Kp^V_\theta(t, T)N[d_2(K)]$$

with

$$d_2(K) = \left(\log\left(\frac{S(t)}{Kp^V_\theta(t, T)}\right) - \frac{1}{2}\Omega^2(t, T)\right)/\left(\sqrt{\Omega^2(t, T)}\right)$$

$$d_1(K) = d_2(K) + \sqrt{\Omega^2(t, T)}$$

$$\Omega^2(t, T) = (T - t)((\sigma(s))^2 + (\sigma(r))^2 + \frac{2\rho\sigma(s)\sigma(r)}{k^2})$$

$$+ (e^{-k(T-t)} - 1)\left(\frac{2(\sigma(r))^2}{k^3} + \frac{2\rho\sigma(s)\sigma(r)}{k^2}\right)$$

$$- \frac{(\sigma(r))^2}{2k^3}(e^{-2k(T-t)} - 1).$$
For $\epsilon$ small enough it holds that:

1. let $p_i(t, T)$ denote the bond price (with $X(t)$ being in the state $i$ at time $t$) then

   \[ p_{g_2}^V(t, T) \geq p_2(t; T) \geq p_1(t; T) \geq p_{g_1}^V(t, T); \]

2. let $c_i(K) \equiv c_i(t, T; K)$, then

   (a) if $X(t) = 1$ then $c_{g_1}^{VM} \geq c_1$;

   (b) if $X(t) = 2$ then $c_{g_2}^{VM} \leq c_2$ if and only if

   \[ S(t)N[d_1]E[Y_i'(0)] - Kp_{g_i}^V(t, T)N[d_2]\bar{\gamma}_i \geq 0 \]